Lecture 20: Pseudorandom Functions

Pseudo-random Functions (PRF)

- Let $\mathcal{G}_{m,n,k} = \{g_1, g_2, \ldots, g_{2^k}\}$ be a set of functions such that each $g_i \colon \{0,1\}^m \to \{0,1\}^n$
- This set of functions $\mathcal{G}_{m,n,k}$ is called a pseudo-random function if the following holds.
  Suppose we pick $g \xleftarrow{\$} \mathcal{G}_{m,n,k}$. Let $x_1, \ldots, x_t \in \{0,1\}^m$ be distinct inputs. Given $(x_1, g(x_1)), \ldots, (x_{t-1}, g(x_{t-1}))$ for any computationally bounded party the value $g(x_t)$ appears to be uniformly random over $\{0,1\}^n$

Secret-key Encryption using Pseudo-Random Functions

Before we construct a PRF, let us consider the following secret-key encryption scheme.

1. $\mathrm{Gen}()$: Return $\mathrm{sk} = \mathrm{id} \xleftarrow{\$} \{1, \ldots, 2^k\}$
2. $\mathrm{Enc}_{\mathrm{id}}(m)$: Pick a random $r \xleftarrow{\$} \{0,1\}^m$. Return $(m \oplus g_{\mathrm{id}}(r), r)$, where $m \in \{0,1\}^n$.
3. $\mathrm{Dec}_{\mathrm{id}}(\widetilde{c}, \widetilde{r})$: Return $\widetilde{c} \oplus g_{\mathrm{id}}(\widetilde{r})$.

Features. Suppose the messages $m_1, \ldots, m_u$ are encrypted as the cipher-texts $(c_1, r_1), \ldots, (c_u, r_u)$.

- As long as the $r_1, \ldots, r_u$ are all distinct, each one-time pad $g_{\mathrm{id}}(r_1), \ldots, g_{\mathrm{id}}(r_u)$ appear uniform and independent of others to computationally bounded adversaries. So, this encryption scheme is secure against computationally bounded adversaries!
- The probability that any two of the randomness in $r_1, \ldots, r_u$ are not distinct is very small (We shall prove this later as "Birthday Paradox")
- This scheme is a "state-less" encryption scheme. Alice and Bob do not need to remember any private state (except the secret-key sk)!

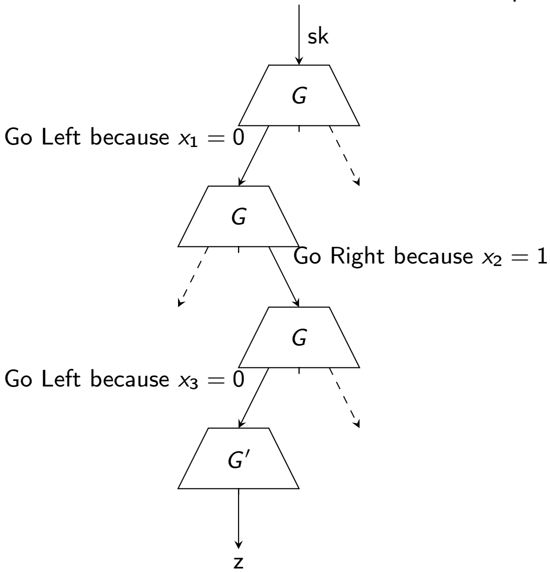
Construction of PRF I

- We shall consider the construction of Goldreich-Goldwasser-Micali (GGM) construction.
- Let $G\colon \{0,1\}^k \to \{0,1\}^{2k}$ be a PRG. We define $G(x) = (G_0(x), G_1(x))$, where $G_0, G_1\colon \{0,1\}^k \to \{0,1\}^k$
- Let $G'\colon \{0,1\}^k \to \{0,1\}^n$ be a PRG
- We define $g_{\mathrm{id}}(x_1x_2\dots x_m)$ as follows

$$G'\left(G_{x_m}(\cdots G_{x_2}(G_{x_1}(\mathrm{id}))\cdots)\right)$$

Construction of PRF II

Consider the execution for $x = x_1x_2x_3 = 010$. Output z is computed as follows.



We give the pseudocode of algorithms to construct PRG and PRF using a OWP $f\colon \{0,1\}^{k/2} \to \{0,1\}^{k/2}$

- Suppose $f\colon \{0,1\}^{k/2} \to \{0,1\}^{k/2}$ is a OWP
- We provide the pseudocode of a PRG $G\colon \{0,1\}^{k} \to \{0,1\}^{t}$, for any integer t, using the one-bit extension PRG construction of Goldreich-Levin hardcore predicate construction. Given input $s \in \{0,1\}^{k}$, it outputs $G(s)$.

$G(k,t,s)$:

1. Interpret $s = (r,x)$, where $r,x \in \{0,1\}^{k/2}$
2. Initialize bits = [ ] (i.e., an empty list)
3. Initialize $z = x$
4. For $i = 1$ to t:
   1. bits.append($\langle r,z\rangle$), here $\langle\cdot,\cdot\rangle$ is the inner-product
   2. $z = f(z)$
5. Return bits

- We provide the pseudocode of the PRF $g_{\mathrm{id}}\colon \{0,1\}^m \to \{0,1\}^n$, where $\mathrm{id} \in \{0,1\}^k$, using the GGM construction. Given input $x \in \{0,1\}^m$, it outputs $g_{\mathrm{id}}(x)$.

$g(m, n, k, \mathrm{id}, x)$:

1. Interpret $x = x_1x_2\dots x_m$, where $x_1, \dots, x_m \in \{0,1\}$
2. Initialize $\mathrm{inp} = \mathrm{id}$
3. For $i = 1$ to m:
    1. Let $y = G(k, 2k, \mathrm{inp})$
    2. If $x_i = 0$, then inp is the first k bits of y. Otherwise (if $x_i = 1$), inp is the last k bits of y.
4. Return $G(k, n, \mathrm{inp})$

- This function evaluation is parallelizable, and its security is based on the "Decisional Diffie-Hellman problem" (DDH)
- Let p and ℓ be prime numbers such that ℓ divides $(p-1)$.
- $g \in F_p^*$ generate a subgroup of order ℓ
- Naor-Reingold PRF are functions $\{0,1\}^n \to F_p^*$ defined below:

$$f_a(x) := g^{a_0 \cdot a_i^{x_1} \cdot a_2^{x_2} \cdots a_n^{x_n}},$$

where

$$a = a_0 a_1 \cdots a_n \in (F_\ell)^{n+1}$$
$$x = x_1 x_2 \cdots x_n \in \{0,1\}^n.$$

- **Note:** For an additive group like an elliptic curve, the definition of the function is

$$f_a(x) := \left(a_0 \cdot a_1^{x_1} \cdot a_2^{x_2} \cdots a_n^{x_n}\right) \cdot G,$$

where the subgroup generated by G has order ℓ.